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STRESS ANALYSIS OF A MATRIX-CRACKED VISCOELASTIC LAMINATE

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Abstract—Two distinctly different approaches to viscoelastic stress analysis are employed herein for the purpose of predicting the response of a matrix-cracked viscoelastic laminate to a given loading history. A viscoelastic correspondence principle is developed to provide an analytical solution and a finite element formulation is developed to provide a numerical solution. The two methods are demonstrated through the solution of a simple illustrative example problem. Results from the two methods of analysis are compared. © 1997 Elsevier Science Ltd.

INTRODUCTION

An effort is now underway to adapt the use of lightweight polymeric composites to applications that involve sustained operation at elevated temperature. The proposed High Speed Civil Transport (HSCT) represents one such application. Design goals for the HSCT include a cruise of Mach 2.4, a range of at least 9250 km, and a lifetime exceeding 60,000 hours. Sustained operation at Mach 2.4 will result in skin temperatures near 200°C. In this environment, polymeric composites are likely to exhibit history dependent viscoelastic behavior. It is also likely that microstructural damage, such as matrix cracking and delamination, will occur as an aircraft ages. Consequently, in order to predict the integrity of HSCT structure, one must be prepared to assess the effect of damage in a viscoelastic composite. It is in response to this need that the current work has been undertaken. We concentrate here on one form of damage only: matrix cracking.

The issue of matrix cracking in laminated composite materials has received considerable attention in recent years, with much of the effort being directed toward an assessment of stiffness reduction. This reduction of stiffness is of considerable importance since it leads to a redistribution of stress within the laminate and an overall nonlinear response. Moreover, it is this redistribution of stress along with damage induced stress concentrations that leads to more damage. Several methods have been proposed for the prediction of stiffness reduction due to matrix cracking. The simplest of these is the so-called ply discount scheme. In this method, whenever a transverse crack appears in a ply, the stiffness in this direction either vanishes or is reduced by a predetermined amount. While simple in both concept and implementation, this method can be rather inaccurate. Other more sophisticated methods that have been applied to this problem include: shear lag analysis, self consistent scheme, energy methods, and the finite element method.

Reifsnider *et al.* (1979) found, through experiment, that matrix cracking in transverse plies will, after sufficient loading, reach a saturation level wherein the cracks will be approximately equally spaced. They referred to this saturation state as the Characteristic Damage State (CDS). Once the CDS is reached, further load induced damage will occur in modes other than transverse cracking. Reifsnider (1977) found that the CDS can be predicted by shear lag analysis. Highsmith and Reifsnider (1982) extended this analysis to include prediction of the effects of matrix cracking on laminate stiffness. Lim and Hong (1989) have extended the method to include an interlaminar shear layer. The effect of thermal residual stresses and Poisson's ratio mismatch are also accounted for in Lim and Hong. Flaggs (1985) has developed a shear lag model that incorporates applied shear as well as cross-ply tension. Shear lag analysis, which is used to ascertain the state of stress, can be used in conjunction with some failure criterion to predict crack initiation or an increase in crack density (i.e., an increase in the number of matrix cracks). Shear lag analysis has been used in conjunction with linear elastic fracture mechanics by Garrett and Bailey (1977), Parvizi *et al.* (1978), Flaggs (1985), Lim and Hong (1989), and Han and Hahn (1989). Peters (1984) and Swanson (1989) have combined shear lag analysis with a Weibull failure criterion to predict further cracking. Lee and Daniel (1990) employed a shear lag model in conjunction with a max-stress failure criterion to the problem of progressive matrix cracking of a cross-ply laminate subjected to tensile loading. This work has been extended by Tsai and Daniel (1991) and by Tsai *et al.* (1990) to consider simple shear and biaxial loading, respectively. While the shear lag model has been shown to yield good results under certain circumstances, it is limited by certain restrictions. In the model, load is transferred between plies containing matrix cracks through shear layers. A shortcoming of the model is that it provides no systematic formulation for assigning either thickness or material properties to these layers. It also fails to account for the variation of stress or strain through the thickness direction of a ply. Another difficulty is that it is not clear how to apply the model to angle-ply laminates.

Dvorak and Laws (1984), Dvorak *et al.* (1983), (1985), and Laws *et al.* (1983) have used the self-consistent method for estimating stiffness reduction due to matrix cracking. In this method, the degradation in elastic properties for a lamina is derived from the degradation in an unbounded cracked medium. Classical lamination theory is then applied with equivalent homogeneous plies to obtain the stiffness reduction of the laminate. Implementation of this scheme is simple and predicted stiffness agrees well with experiment. This method cannot, however, be used to predict damage evolution.

Several approximate analytical solutions based on various energy methods have been proposed. An attractive feature of these methods is that they tend to produce either an upper or lower bound on reduced stiffness. Aboudi (1987) has developed an approximate analytical solution for predicting reduced stiffness in a cracked body. In this analysis, the displacement field in a unit cell or Representative Volume Element (RVE) with aligned cracks is expanded in Legendre polynomials. The equilibrium equations, in conjunction with continuity requirements on displacements and tractions at RVE interfaces, leads to a set of linear ordinary differential equations in the unknown field variables. A finite differencing procedure reduces these to a set of algebraic equations. Upon solution of these, degraded stiffness is determined from a calculation of the strain energy in the cracked body. The method has been applied to cracked isotropic solids, unidirectional and cross-ply laminates. This method produces an upper bound on reduced stiffness. The method has been extended by Herakovich *et al.* (1988) in an investigation of the effects of matrix cracking in cross-ply laminates. In this work, degradation of shear modulus, Poisson's ratio, and coefficients of thermal expansion are predicted in addition to the degradation of axial modulus. It is shown that shear modulus and Poisson's ratio exhibit significantly greater degradation than axial modulus for corresponding crack densities. The method is further extended in Aboudi *et al.* (1988) in an investigation of the three dimensional problem of cross-cracks in a composite laminate. Lee (1990), has also developed an approximate analytical method which produces an upper bound on reduced stiffness. This method, like Aboudi's, employs an assumed displacement field which is consistent with certain assumptions regarding a representative volume element. Minimization of potential energy leads to an analytical expression for reduced axial and shear moduli. Of particular significance is the observation that application of this method is not restricted to cross-ply laminates. In addition, the method can be used in conjunction with a failure criterion to analyze progressive matrix cracking. Lee *et al.* (1989), (1991) have applied this method to the problem of predicting stiffness reduction in cross-ply laminates containing transverse matrix cracks. Allen and Lee (1991) applied this approach to angle-ply as well as cross-ply laminates and considered cyclic as well as monotonic loading. Hashin (1985), (1987), (1989) has developed a method based on minimization of complimentary energy which produces a lower bound for reduced stiffness. In this method, Hashin constructs admissible stress fields which satisfy equilibrium and all boundary and interface conditions. This requirement makes the method difficult to apply to all but the simplest forms of cross-ply laminates i.e.,

$[0_q/90_r]_s$. The method has been used to derive expressions for coefficients of thermal expansion as well as elastic moduli in matrix-cracked cross-ply laminates. Through the use of more extensive trial stress functions, Varna and Berglund (1991) have made some improvements to the Hashin model. Nairn (1989) has extended Hashin's model to include a spatially homogeneous temperature change and to provide an explicit analytical expression for energy release rate.

Gudmundson and Östlund (1992a) have shown that for dilute matrix crack densities (dilute implying no interaction between cracks) closed form expressions for laminate stiffnesses can be derived which are asymptotically exact as the crack density goes to zero. The theory is based on the hypothesis that if an expression for the elastic strain energy in terms of stresses or strains of a cracked material can be derived, then the stiffness and compliance tensors can be determined from a simple identification. The theory is extended in Gudmundson and Östlund (1992b) to include estimations of the reduction in thermal expansion coefficients and predictions of average strains resulting from release of residual stresses. The authors also introduce an alternative model which is asymptotically exact as crack density goes to infinity. Gudmundson *et al.* (1992) further extend the theory to include prediction of average ply stresses in a microcracked laminate.

Nuismer and Tan (1983) have presented a matrix cracking model based on fracture mechanics and an approximate elasticity solution derived from the work of Hegemier *et al.* (1973) and Nayfeh (1977). This approximate elasticity theory is used by Nuismer and Tan (1988) to derive cracked lamina constitutive equations. Compliances are given for general in-plane loading including the effects of non-mechanical strains and are written explicitly in typical laminated plate form. It is shown that the damaged ply constitutive relations are not independent of laminate stacking sequence. Tan and Nuismer (1989) extend the theory to the modeling of progressive matrix cracking of laminates containing a cracked 90° ply and subjected to tensile or shear loading. Closed form solutions are obtained for laminate stiffnesses and Poisson's ratio as a function of crack density or load level. A significant improvement to the theory of Gudmundson and Östlund (1992a), (1992b), and of Gudmundson *et al.* (1992), which is valid for intermediate crack densities, has been presented by Gudmundson and Zang (1993). The key to this improvement is the use of an analytical elasticity solution for a row of cracks in an infinite, homogeneous, isotropic medium (Bentham and Koiter (1972) and Tada *et al.* (1973)). Kaw and Besterfield (1992) have presented an elasticity solution to the problem of periodic, interacting and regularly spaced matrix cracks in a unidirectional fiber-reinforced brittle matrix composite. The solution is obtained in terms of a hyper-singular integral equation. The effects of fiber reinforcement, and the spacing, the location and the length of cracks on the stress intensity factors at the crack tips and the maximum crack opening displacement in the composite are studied. The theory is extended in Kaw and Gadi (1992) to include variable fiber volume fraction. The stress intensity factors at the crack tips, the longitudinal stiffness of the composite and the matrix initiation stress are studied as functions of the elastic moduli of the constituents, the fiber volume fraction, the transverse crack spacing, and the crack length.

Methods based on continuum damage mechanics and internal state variables have been proposed by Talreja (1985a), (1985b) and by Allen *et al.* (1987a), (1987b). Finite element methods have been applied to the problem of matrix cracking by Herakovich *et al.* (1988), Gudmundson and Östlund (1992a), (1992c), Gudmundson *et al.* (1992), Gudmundson and Zang (1993), and Groves *et al.* (1987).

This brief review of the literature attests to the fact that a wealth of information already exists on the topic of matrix cracking. It should be recognized, however, that in virtually all of the literature now available the analyses were conducted assuming the body to be elastic. To the knowledge of the authors, only the works of Zocher *et al.* (1994), (1995) and of Schapery and Sicking (1995) deal with the problem of matrix cracking in a viscoelastic laminate. Schapery and Sicking (1995) use a work potential to account for nonlinear elastic behavior of the fibers, inelastic behavior of the matrix, transverse cracking, and delamination. The present work represents the completion of the analysis begun in Zocher *et al.* (1994), (1995).

In this research, we address the problem of matrix cracking in a viscoelastic laminate.

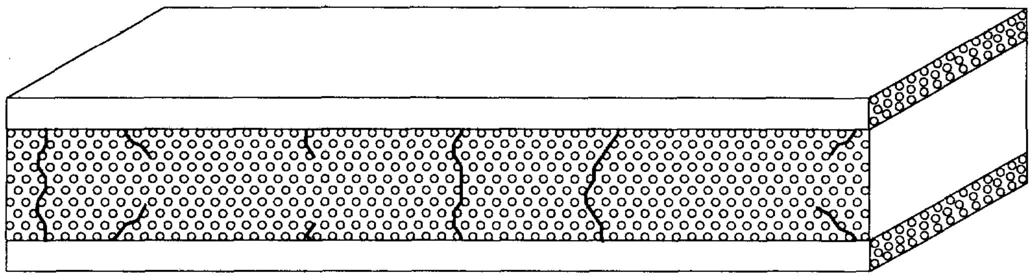

Fig. 1. Typical laminate with matrix cracks.

We do so through the presentation of an investigation of the response of a matrix-cracked laminate to a given loading history. This investigation is conducted through the implementation of two distinctly different approaches to viscoelastic stress analysis. A viscoelastic Correspondence Principle (CP) is developed to provide a purely analytical prediction, while the finite element method is used for a numerical prediction. Results from the two methods of analysis are compared. The development of the correspondence principle represents an extension of the elastic analyses presented by Lee and coworkers (Lee (1990), Lee *et al.* (1989), (1991), and Allen and Lee (1991)). In the following, we shall present the analytical approach first, followed by the numerical work.

ANALYTICAL ANALYSIS

In order for the reader to understand the extension of the analyses of Lee *et al.* (Lee (1990), Lee *et al.* (1989), (1991), and Allen and Lee (1991)) to the viscoelastic realm, it will be necessary for us to first describe their elastic analysis. We do so only to the degree necessary to enable the reader to understand the extension. Much of what is to be presented in our discussion of the elastic analysis is not found explicitly in any of the papers of Lee *et al.* (Lee (1990), Lee *et al.* (1989), (1991), and Allen and Lee (1991)). It is also noted that Lee *et al.* addressed some topics such as failure analysis that are not directly addressed here. Once we have presented an outline of the elastic analysis, we will proceed with the extension of the analysis to account for viscoelastic behavior.

Elastic analysis

Consider a matrix-cracked cross-ply laminate such as that shown in Fig. 1 possessing an arbitrary number of 0° and 90° layers, subjected to in-plane loading. The pattern of matrix cracking that might develop in such a laminate is (as depicted in Fig. 1) rather complex; involving cracks of varying shape, length, and orientation. In order to render the analysis tractable, Lee and coworkers chose to approximate the damage with an array of equally spaced and regularly shaped matrix cracks as shown on the left in Fig. 2. They could then consider a statistically averaged RVE which, when subjected to a kinematically admissible displacement field, enabled the derivation of homogenized properties for an equivalent uncracked layer. The RVE is shown on the right in Fig. 2. By assuming the

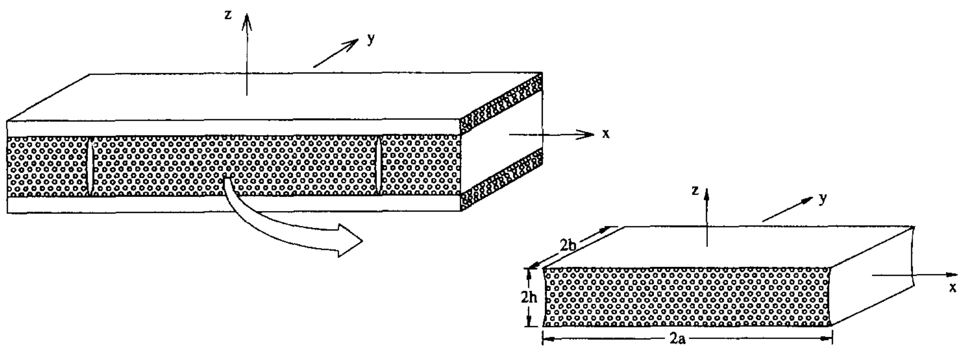

Fig. 2. Assumed crack pattern and RVE.

crack surfaces to be frictionless, the inplane biaxial tension and inplane shear problems uncouple. It is noted that the solution of Lee and coworkers is independent of stacking sequence and can be used for angle-ply as well as cross-ply laminates.

Lee and coworkers assumed each uncracked ply to be transversely isotropic with independent material constants E_L, E_T, ν_{LT}, ν_T, and G_{LT}. The ply level constitutive properties C_L, C_T, C_{LT}, C_{23}, and G_T are then calculated from the following as given by Whitney (1987).

$$C_L = (1-\nu_T^2)\frac{E_L}{V} \qquad C_{LT} = \nu_{LT}(1+\nu_T)\frac{E_T}{V}$$

$$C_T = \left(1-\nu_{LT}^2\frac{E_T}{E_L}\right)\frac{E_T}{V} \qquad C_{23} = \left(\nu_T+\nu_{LT}^2\frac{E_T}{E_L}\right)\frac{E_T}{V}$$

$$G_T = \frac{E_T}{2(1+\nu_T)} \qquad V = (1+\nu_T)\left(1-\nu_T-2\nu_{LT}^2\frac{E_T}{E_L}\right). \tag{1}$$

The displacement field that Lee and coworkers imposed on the RVE for the problem of biaxial tension (which can be shown to satisfy equilibrium in the x and y directions) is given by:

$$\begin{aligned} u &= \left(\frac{u_0}{a}\right)x + \sum_{n=1}^{\infty} C_n \sinh\alpha_n x \cos\gamma_n z \\ v &= \left(\frac{v_0}{b}\right)y \\ w &= \left(\frac{w_0}{h}\right)z \end{aligned} \tag{2}$$

where

$$\alpha_n = \sqrt{\frac{G_T}{C_T}}\gamma_n \qquad \gamma_n = \frac{(2n-1)\pi}{2h}. \tag{3}$$

In the above, u_0, v_0, and w_0 are constants and the term C_n will be defined momentarily. The reader will perhaps recognize that eqn (2) represents a Ritz approximation. Hence the particular form of the equations (i.e., the use of sinh and cos) is not derivable, it is merely a "guess" in the sense of the Ritz method. Requirements on the selection of Ritz basis functions are given by Reddy (1984). It follows directly from the displacement field that the strains and stresses are given by:

$$\begin{aligned} \varepsilon_{xx} &= \left(\frac{u_0}{a}\right) + \sum_{n=1}^{\infty} C_n\alpha_n \cosh\alpha_n x \cos\gamma_n z \\ \varepsilon_{yy} &= \left(\frac{v_0}{b}\right) \\ \varepsilon_{zz} &= \left(\frac{w_0}{h}\right) \\ 2\varepsilon_{yz} &= 0 \\ 2\varepsilon_{xz} &= -\sum_{n=1}^{\infty} C_n\gamma_n \sinh\alpha_n x \sin\gamma_n z \\ 2\varepsilon_{xy} &= 0 \end{aligned} \tag{4}$$

and

$$\sigma_{xx} = C_T(e_1) + C_{LT}(e_2) + C_{23}(e_3) + C_T \sum_{n=1}^{\infty} A_n(x, z)$$

$$\sigma_{yy} = C_{LT}(e_1) + C_L(e_2) + C_{LT}(e_3) + C_{LT} \sum_{n=1}^{\infty} A_n(x, z)$$

$$\sigma_{zz} = C_{23}(e_1) + C_{LT}(e_2) + C_T(e_3) + C_{23} \sum_{n=1}^{\infty} A_n(x, z)$$

$$\sigma_{yz} = 0$$

$$\sigma_{xz} = -G_T \sum_{n=1}^{\infty} C_n \gamma_n \sinh \alpha_n x \sin \gamma_n z$$

$$\sigma_{xy} = 0 \tag{5}$$

where

$$A_n(x, z) = C_n \alpha_n \cosh \alpha_n x \cos \gamma_n z$$

and

$$e_1 = \frac{u_0}{a} - \beta_T \Theta$$

$$e_2 = \frac{v_0}{b} - \beta_L \Theta$$

$$e_3 = \frac{w_0}{h} - \beta_T \Theta. \tag{6}$$

The β's in eqn (6) are coefficients of thermal expansion and Θ is the difference between the current temperature and a stress-free reference temperature. Imposing the boundary condition $\sigma_{xx_{|x=a}} = 0$ gives:

$$C_n = \frac{(-2)(-1)^{n+1}[C_T(e_1) + C_{LT}(e_2) + C_{23}(e_3)]}{C_T h \alpha_n \gamma_n \cosh \alpha_n a}. \tag{7}$$

Although straightforward, the derivation of eqn (7) is far too lengthy to be presented explicitly here. The key step in the derivation is an exploitation of the orthogonality of $\cos(f\pi z/g)$ ($f = 2n-1$, and $g = 2h$). The outline is now complete except for one important detail; how do we choose u_0, v_0, and w_0? The answer to this question is developed in the next few pages. The first step toward finding that answer is to determine the strain energy in the RVE. This may be calculated from the following:

$$U = \frac{1}{2} \int_{V_L} C_{ijkl} \varepsilon_{ij} (\varepsilon_{kl} - 2\beta_{kl} \Theta) \, \mathrm{d}V. \tag{8}$$

Here C_{ijkl} is a fourth order tensor of elastic constants. Solving eqn (8) yields the following expression for the strain energy density (strain energy per unit volume) of the RVE:

$$U_0 = \tfrac{1}{2}\{-\lambda[C_T e_1 + C_{LT} e_2 + C_{23} e_3]^2 + C_T(e_1^2 + e_3^2) + C_L e_2^2 + 2C_{LT}(e_1 e_2 + e_2 e_3) + 2C_{23} e_1 e_3\} - \tfrac{1}{2} C_{ijkl}\beta_{ij}\beta_{kl}\Theta^2 \quad (9)$$

where

$$\lambda = \frac{16}{C_T \pi^3} \sum_{m=1}^{\infty} \frac{\tanh\left\{\dfrac{(2m-1)\pi a}{2h}\sqrt{\dfrac{G_T}{C_T}}\right\}}{(2m-1)^3 \dfrac{a}{h}\sqrt{\dfrac{G_T}{C_T}}}. \quad (10)$$

Now since u_0/a, v_0/b, and w_0/h represent components of the boundary averaged strain (ϵ_x, ϵ_y, and ϵ_z, respectively), the following partial derivatives provide us with expressions for the volume averaged stress in the RVE.

$$\frac{\partial U_0}{\partial\left(\dfrac{u_0}{a}\right)}, \quad \frac{\partial U_0}{\partial\left(\dfrac{v_0}{b}\right)}, \quad \frac{\partial U_0}{\partial\left(\dfrac{w_0}{h}\right)}.$$

Determining these partial derivatives (using the notation "$\langle$*argument*$\rangle$" to denote the volume average of the *argument*) yields:

$$\begin{Bmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \\ \langle\sigma_z\rangle \end{Bmatrix} = \begin{bmatrix} C_T - C_T^2\lambda & C_{LT} - C_{LT}C_T\lambda & C_{23} - C_{23}C_T\lambda \\ C_{LT} - C_{LT}C_T\lambda & C_L - C_{LT}^2\lambda & C_{LT} - C_{23}C_{LT}\lambda \\ C_{23} - C_{23}C_T\lambda & C_{LT} - C_{23}C_{LT}\lambda & C_T - C_{23}^2\lambda \end{bmatrix} \begin{Bmatrix} \dfrac{u_0}{a} - \beta_T\Theta \\ \dfrac{v_0}{b} - \beta_L\Theta \\ \dfrac{w_0}{h} - \beta_T\Theta \end{Bmatrix}. \quad (11)$$

It is noted that an alternative (and admittedly more direct) calculation of $\langle\sigma_{ij}\rangle$ can be accomplished through a direct application of the definition for $\langle\sigma_{ij}\rangle$ operating on eqn (5), namely:

$$\langle\sigma_{ij}\rangle = \frac{1}{V_L}\int_{V_L} \sigma_{ij}\,\mathrm{d}V. \quad (12)$$

Suppose for the moment, that we treat u_0/a and v_0/b as known quantities. The total potential energy, Π, in the RVE then becomes a function of w_0/h only. Minimization of the total potential energy then requires that the partial derivative of Π with respect to w_0/h be zero. This in turn requires that the z-component of the volume averaged stress be identically zero, that is:

$$\delta\Pi = 0 \Rightarrow \frac{\partial\Pi}{\partial\left(\dfrac{w_0}{h}\right)} = 0 \Rightarrow \langle\sigma_z\rangle = 0.$$

Now, substituting $\langle\sigma_z\rangle = 0$ into eqn (11) yields the following expression for w_0/h:

$$\frac{w_0}{h} = \frac{(C_{23}C_T\lambda - C_{23})\left(\frac{u_0}{a} - \beta_T\Theta\right) + (C_{23}C_{LT}\lambda - C_{LT})\left(\frac{v_0}{b} - \beta_L\Theta\right)}{C_T - \lambda C_{23}^2} + \beta_T\Theta. \tag{13}$$

Then using this expression for w_0/h in the first two equations of eqn (11) leads to the following set of equations for the x- and y-components of the volume averaged stress.

$$\begin{Bmatrix} \langle\sigma_x\rangle \\ \langle\sigma_y\rangle \end{Bmatrix} = \begin{bmatrix} Q_T(1-\zeta) & Q_{LT}(1-\zeta) \\ Q_{LT}(1-\zeta) & Q_L(1-\nu_{LT}\nu_{TL}\zeta) \end{bmatrix} \begin{Bmatrix} u_0/a - \beta_T\Theta \\ v_0/b - \beta_L\Theta \end{Bmatrix} \tag{14}$$

where

$$\begin{aligned} Q_L &= C_L - \frac{C_{LT}^2}{C_T} \\ Q_{LT} &= C_{LT} - \frac{C_{23}C_{LT}}{C_T} \\ Q_T &= C_T - \frac{C_{23}^2}{C_T} \end{aligned} \tag{15}$$

and

$$\zeta = \lambda \frac{C_T^2 - C_{23}^2}{C_T - \lambda C_{23}^2} = \frac{\lambda C_T(1+\nu_T)(1-\nu_T - 2\nu_{LT}\nu_{TL})}{(1-\nu_{LT}\nu_{TL})^2 - \lambda C_T(\nu_T + \nu_{LT}\nu_{TL})^2}. \tag{16}$$

The term ζ is a damage parameter for biaxial tension. Recall that the biaxial tension and shear loading problems are uncoupled in this analysis. If we now add the shear terms, similarly derived, to the set of equations given in eqn (14), we complete the damage dependent constitution for a linear elastic ply containing matrix cracks. The effective constitutive relations for a matrix-cracked ply are then given as:

$$\begin{Bmatrix} \langle\sigma_L\rangle \\ \langle\sigma_T\rangle \\ \langle\sigma_{LT}\rangle \end{Bmatrix} = \begin{bmatrix} Q_L(1-\nu_{LT}\nu_{TL}\zeta) & Q_{LT}(1-\zeta) & 0 \\ Q_{LT}(1-\zeta) & Q_T(1-\zeta) & 0 \\ 0 & 0 & G_{LT}(1-\phi) \end{bmatrix} \begin{Bmatrix} \epsilon_L - \beta_L\Theta \\ \epsilon_T - \beta_T\Theta \\ 2\epsilon_{LT} \end{Bmatrix}. \tag{17}$$

The term ϕ appearing in the above is a shearing mode damage parameter similar to ζ. Since it is not used in this paper, we will not go into its derivation nor provide a formal definition of it; the reader is referred to Lee *et al.* (Lee (1990), Lee *et al.* (1989), (1991), and Allen and Lee (1991)) for the development of ϕ. The third equation of eqn (17) is included only for completeness of the statement. Note that this set of equations corresponds exactly to the set of equations for lamina constitution in terms of the reduced stiffness matrix as given in any text on the mechanics of composites (i.e., Jones (1975) p. 46), namely $\{\sigma\} = [Q]\{\varepsilon\}$. The "reduced stiffness matrix" is of course different here but its use is precisely the same. For example, eqn (17) can be used in classical lamination theory exactly as $\{\sigma\} = [Q]\{\varepsilon\}$ would be. For instance, in the calculation of effective laminated plate constitution in terms of force resultants, mid-plane strains, and curvatures:

$$\begin{Bmatrix} N+N^T \\ \cdots \\ M+M^T \end{Bmatrix} = \begin{bmatrix} A & \vdots & B \\ \cdots & \cdots & \cdots \\ B & \vdots & D \end{bmatrix} \begin{Bmatrix} \varepsilon_0 \\ \cdots \\ \kappa \end{Bmatrix}. \tag{18}$$

It is noted that eqn (17) reduces in the undamaged case ($\zeta = \phi = 0$) to $\{\sigma\} = [Q]\{\varepsilon\}$ as given by Jones (1975).

Now that we have established the effective ply-level constitutive relations for a ply containing matrix cracks, let us consider the application of what has been developed to a particular class of laminates. Let us consider laminates of the form $[0_q/90_r]_s$. Applying what has just been presented to this class of laminates produces the following set of equations for laminated plate constitution:

$$\begin{Bmatrix} N_x+N_x^T \\ N_y+N_y^T \end{Bmatrix} = 2PPT \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{u_0}{a} \\ \dfrac{v_0}{b} \end{Bmatrix}. \tag{19}$$

Here PPT stands for per-ply-thickness and the H's are defined as follows:

$$\begin{aligned} H_{11} &= qQ_L + r(1-\zeta)Q_T \\ H_{12} &= [q + r(1-\zeta)]Q_{LT} \\ H_{22} &= r(1-\nu_{LT}\nu_{TL}\zeta)Q_L + qQ_T. \end{aligned} \tag{20}$$

Recall that N_x, N_y, etc., appearing in eqn (19) are force resultants with the units of force per length. If we multiply each side of eqn (19) by one over the plate thickness, then we can equivalently express these equations in terms of stress resultants or surface tractions as given below:

$$\begin{Bmatrix} \sigma_x^M+\sigma_x^T \\ \sigma_y^M+\sigma_y^T \end{Bmatrix} = \frac{1}{q+r} \begin{bmatrix} H_{11} & H_{12} \\ H_{12} & H_{22} \end{bmatrix} \begin{Bmatrix} \dfrac{u_0}{a} \\ \dfrac{v_0}{b} \end{Bmatrix}. \tag{21}$$

Consider the case of uniaxial loading of the matrix-cracked plate. We shall denote the surface traction in the x-direction as σ_x^{LAM}; the y-component is zero. Under this loading condition, eqn (21) simplifies to:

$$\begin{Bmatrix} \sigma_x^{LAM}+\sigma_x^T \\ \sigma_y^T \end{Bmatrix} = \begin{bmatrix} \dfrac{H_{11}}{q+r} & \dfrac{H_{12}}{q+r} \\ \dfrac{H_{12}}{q+r} & \dfrac{H_{22}}{q+r} \end{bmatrix} \begin{Bmatrix} \dfrac{u_0}{a} \\ \dfrac{v_0}{b} \end{Bmatrix}. \tag{22}$$

The above may be inverted to yield the following:

$$\begin{Bmatrix} \dfrac{u_0}{a} \\ \dfrac{v_0}{b} \end{Bmatrix} = \begin{bmatrix} \dfrac{H_{11}}{q+r} & \dfrac{H_{12}}{q+r} \\ \dfrac{H_{12}}{q+r} & \dfrac{H_{22}}{q+r} \end{bmatrix}^{-1} \begin{Bmatrix} \sigma_x^{LAM} + \sigma_x^T \\ \sigma_y^T \end{Bmatrix}. \tag{23}$$

Now substituting in expressions for the thermal components, σ_x^T and σ_y^T (these are calculated in a manner analogous to that used in classical lamination theory) and expanding (also repeating eqn (13)) we produce the following:

$$\frac{u_0}{a} = \frac{\sigma_x^{LAM}}{E_x^c} - \frac{q(\beta_T - \beta_L)\Theta}{(q+r)E_x^c}\left[Q_L - Q_{LT} + \frac{H_{12}}{H_{22}}(Q_T - Q_{LT})\right] + \beta_T\Theta$$

$$\frac{v_0}{b} = \frac{-H_{12}\sigma_x^{LAM}}{H_{22}E_x^c} + \frac{q(\beta_T - \beta_L)\Theta}{(q+r)E_x^c H_{22}}[H_{12}(Q_L - Q_{LT}) + H_{11}(Q_T - Q_{LT})] + \beta_L\Theta$$

$$\frac{w_0}{h} = \frac{(C_{23}C_T\lambda - C_{23})\left(\dfrac{u_0}{a} - \beta_T\Theta\right) + (C_{23}C_{LT}\lambda - C_{LT})\left(\dfrac{v_0}{b} - \beta_L\Theta\right)}{C_T - \lambda C_{23}^2} + \beta_T\Theta \tag{24}$$

and we now have a means of determining u_0, v_0, and w_0, at least for a $[0_q/90_r]_s$ laminate with matrix cracks in the 90° layer subjected to a uniaxial surface traction. Of course the development would follow analogously for a different layup or a different loading condition. It can also be shown that for this laminate and this loading condition, the averaged laminate axial modulus is given by the following:

$$E_x^c = \frac{H_{11}H_{22} - H_{12}^2}{H_{22}(q+r)}. \tag{25}$$

Now consider loading this $[0_q/90_r]_s$ laminate by a uniaxial enforced displacement (this is the form of loading that will be applied in the illustrative example problem presented in the following sections). Under this loading condition, v_0/b and w_0/h can be expressed in terms of u_0/a and Θ as follows:

$$\frac{v_0}{b} = \left(-\frac{H_{12}}{H_{22}}\right)\left\{\frac{u_0}{a} - \left[\frac{q(\beta_T - \beta_L)(Q_T - Q_{LT})}{(q+r)H_{12}} + \beta_T\right]\Theta\right\} + \beta_L\Theta$$

$$\frac{w_0}{h} = \frac{(C_{23}C_T\lambda - C_{23})}{C_T - \lambda C_{23}^2}\left(\frac{u_0}{a} - \beta_T\Theta\right)$$
$$+ \frac{(C_{23}C_{LT}\lambda - C_{LT})}{C_T - \lambda C_{23}^2}\left(-\frac{H_{12}}{H_{22}}\right)\left\{\frac{u_0}{a} - \left[\frac{q(\beta_T - \beta_L)(Q_T - Q_{LT})}{(q+r)H_{12}} + \beta_T\right]\Theta\right\} + \beta_T\Theta. \tag{26}$$

Now, substituting these into eqn (5) gives the state of stress in the RVE under a uniaxial enforced displacement. For example, the σ_{xx} component of stress is given by:

$$\sigma_{xx} = \Upsilon\Psi\left(\frac{u_0}{a}\right) + \Omega\Psi\Theta \tag{27}$$

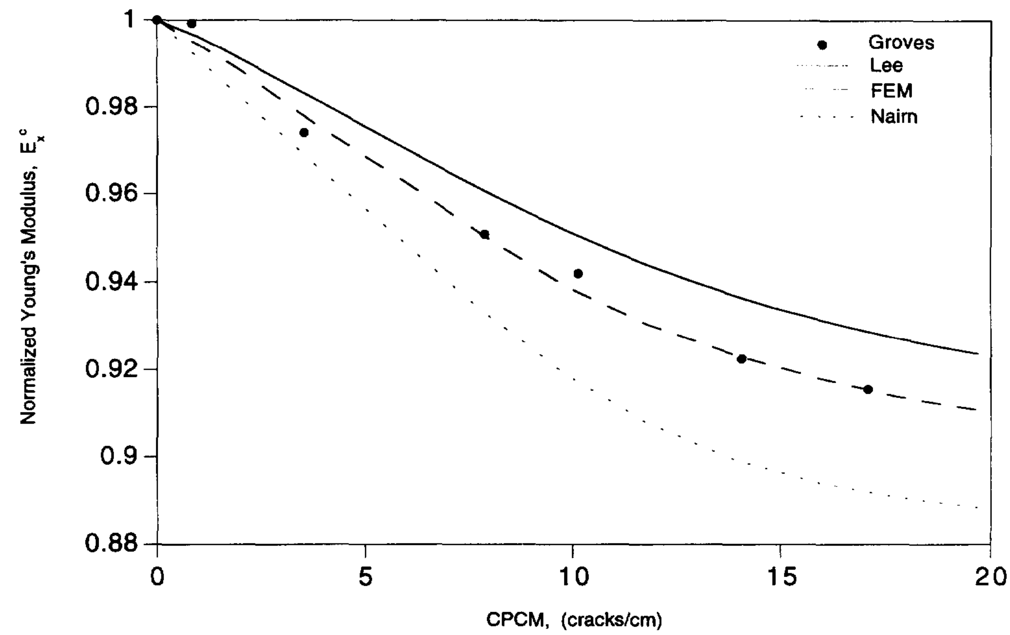

Fig. 3. Normalized Young's modulus, E^c_x, vs crack density.

where

$$\Upsilon = \frac{\zeta}{\lambda} - \frac{C_{LT}H_{12}}{H_{22}} + \frac{C_{LT}H_{11}}{H_{22}} \frac{(C_{23} - \lambda C_{23}^2)}{(C_T - \lambda C_{23}^2)}$$

$$\Psi = 1 + \sum_{n=1}^{\infty} \frac{(-4)(-1)^{n+1}}{(2n-1)\pi} \frac{\cosh \alpha_n x}{\cosh \alpha_n a} \cos \gamma_n z$$

$$\Omega = \frac{\zeta}{\lambda}\left[\frac{q\nu_{LT}(\beta_T - \beta_L)(Q_T - Q_{LT})}{(q+r)H_{22}} + \frac{(\nu_{LT}H_{12} - H_{22})\beta_T}{H_{22}}\right]. \tag{28}$$

This completes our outline of the elastic solution of Lee *et al.* (Lee (1990), Lee *et al.* (1989), (1991), and Allen and Lee (1991)).

An illustration of the use and accuracy of the elastic solution of Lee *et al.*, is provided in Fig. 3. Here the average laminate axial modulus (as predicted by three different methods) is presented as a function of crack density and compared to experimental data. The modulus has been normalized with respect to the uncracked value. The experimental data (filled circles) is from Groves *et al.* (1987) and is for a $[0/90_2]_s$ laminate constructed from AS4/3502 Gr/Ep. The mechanical properties for this material system are given in reference Groves *et al.* (1987). The curve denoted Lee's prediction is generated from eqn (25); Nairn's prediction (Nairn (1989)) is based on the lower bound solution of Hashin (1985), (1987). The finite element prediction was made using an elastic version of ORTHO3D (this program is discussed in the section on numerical analysis). The upper bound nature of the method of Lee *et al.*, and the lower bound nature of Nairn's method (as discussed in the introduction), are apparent in Fig. 3.

Viscoelastic extension

In what is to follow, an over-bar represents the Laplace transform of a variable and an over-tilde represents the Carson transform (s-multiplied Laplace transform) of that variable. The symbol χ is used to denote spatial position. In addition, an over-hat is used to indicate a quantity that is known *a priori*. The governing field and boundary equations for the uncoupled linear thermoviscoelastic Initial/Boundary Value Problem (IBVP) are given by:

$$\sigma_{ji,j}+\rho f_i = 0 \tag{29}$$

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j}+u_{j,i}) \tag{30}$$

$$\sigma_{ij}(\chi, t) = \int_0^t C_{ijkl}(t-\tau, T)\frac{\partial\varepsilon_{kl}(\chi, \tau)}{\partial\tau}\mathrm{d}\tau - \int_0^t \beta_{ij}(t-\tau, T)\frac{\partial\Theta(\chi, \tau)}{\partial\tau}\mathrm{d}\tau \tag{31}$$

$$u_i = \hat{u}_i \quad \text{on} \quad \partial\Omega_1 \tag{32}$$

$$T_i = \sigma_{ji}n_j = \hat{T}_i \quad \text{on} \quad \partial\Omega_2. \tag{33}$$

We note that C_{ijkl} denotes a tensor of relaxation moduli here whereas it denoted a tensor of elastic constants in the earlier section on elastic analysis.

It is easily shown (Schapery (1984), Christensen (1982)) that taking the Laplace transform of this set of equations produces a set of equations which correspond in a one-to-one fashion to the set of equations produced by taking the Laplace transform of the governing field and boundary equations of the linear elastic BVP. The only difference between these two sets of Laplace transformed equations is that $\tilde{C}_{ijkl}$ and $\tilde{\beta}_{ij}$ appear in the viscoelastic set whereas C_{ijkl} and β_{ij} appear in the elastic set. This one-to-one correspondence that exists in Laplace space is the essence of the viscoelastic CP which may be succinctly stated as follows. If one possesses the analytical solution to an elastic problem, and if that solution is Laplace transformable, then one can extract the Laplace transform of the solution to the corresponding viscoelastic problem simply by replacing the elastic material properties in the Laplace transform of the elastic solution with the Carson transform of their viscoelastic counterparts. The viscoelastic solution is then found by taking the inverse Laplace transform. An important restriction on this method is that it can only be used in cases where the essential and natural boundaries are independent of time (i.e., fixed cracks).

Applying the CP to the elastic analysis of Lee *et al.*, as prescribed above, viscoelastic solutions can be obtained. To illustrate, let us consider uniaxial enforced displacement imposed on a $[0_q/90_r]_s$ viscoelastic laminate. By applying the CP to the solution of Lee *et al.*, the Laplace transform of the σ_{xx} component of stress is given by:

$$\bar{\sigma}_{xx} = \tilde{\Upsilon}\tilde{\Psi}\left(\frac{\bar{u}_0}{a}\right)+\tilde{\Omega}\tilde{\Psi}\bar{\Theta} \tag{34}$$

where

$$\begin{aligned}
\tilde{\Upsilon} &= \frac{\tilde{\zeta}}{\tilde{\lambda}}-\frac{\tilde{C}_{LT}\tilde{H}_{12}}{\tilde{H}_{22}}+\frac{\tilde{C}_{LT}\tilde{H}_{11}}{\tilde{H}_{22}}\frac{(\tilde{C}_{23}-\tilde{\lambda}\tilde{C}_{23}^2)}{(\tilde{C}_T-\tilde{\lambda}\tilde{C}_{23}^2)} \\
\tilde{\Psi} &= 1+\sum_{n=1}^{\infty}\frac{(-4)(-1)^{n+1}}{(2n-1)\pi}\frac{\cosh\tilde{\alpha}_n x}{\cosh\tilde{\alpha}_n a}\cos\gamma_n z \\
\tilde{\Omega} &= \frac{\tilde{\zeta}}{\tilde{\lambda}}\left[\frac{q\tilde{v}_{LT}(\tilde{\beta}_T-\tilde{\beta}_L)(\tilde{Q}_T-\tilde{Q}_{LT})}{(q+r)\tilde{H}_{22}}+\frac{(\tilde{v}_{LT}\tilde{H}_{12}-\tilde{H}_{22})\tilde{\beta}_T}{\tilde{H}_{22}}\right]
\end{aligned} \tag{35}$$

$$\tilde{\alpha}_n = \sqrt{\frac{\tilde{G}_T}{\tilde{C}_T}}\gamma_n \tag{36}$$

$$\tilde{C}_L = (1-\tilde{\nu}_T^2)\frac{\tilde{E}_L}{\tilde{V}} \qquad \tilde{C}_{LT} = \tilde{\nu}_{LT}(1+\tilde{\nu}_T)\frac{\tilde{E}_T}{\tilde{V}}$$

$$\tilde{C}_T = \left(1-\tilde{\nu}_{LT}^2\frac{\tilde{E}_T}{\tilde{E}_L}\right)\frac{\tilde{E}_T}{\tilde{V}} \qquad \tilde{C}_{23} = \left(\tilde{\nu}_T+\tilde{\nu}_{LT}^2\frac{\tilde{E}_T}{\tilde{E}_L}\right)\frac{\tilde{E}_T}{\tilde{V}}$$

$$\tilde{G}_T = \frac{\tilde{E}_T}{2(1+\tilde{\nu}_T)} \qquad \tilde{V} = (1+\tilde{\nu}_T)\left(1-\tilde{\nu}_T-2\tilde{\nu}_{LT}^2\frac{\tilde{E}_T}{\tilde{E}_L}\right) \tag{37}$$

$$\tilde{Q}_L = \tilde{C}_L - \frac{\tilde{C}_{LT}^2}{\tilde{C}_T} \qquad \tilde{Q}_{LT} = \tilde{C}_{LT} - \frac{\tilde{C}_{23}\tilde{C}_{LT}}{\tilde{C}_T} \qquad \tilde{Q}_T = \tilde{C}_T - \frac{\tilde{C}_{23}^2}{\tilde{C}_T} \tag{38}$$

$$\begin{aligned}\tilde{H}_{11} &= q\tilde{Q}_L + r(1-\tilde{\zeta})\tilde{Q}_T \\ \tilde{H}_{12} &= [q+r(1-\tilde{\zeta})]\tilde{Q}_{LT} \\ \tilde{H}_{22} &= r(1-\tilde{\nu}_{LT}\tilde{\nu}_{TL}\tilde{\zeta})\tilde{Q}_L + q\tilde{Q}_T\end{aligned} \tag{39}$$

$$\tilde{\lambda} = \frac{16}{\tilde{C}_T\pi^3}\sum_{m=1}^{\infty}\frac{\tanh\left\{\frac{(2m-1)\pi a}{2h}\sqrt{\frac{\tilde{G}_T}{\tilde{C}_T}}\right\}}{(2m-1)^3\frac{a}{h}\sqrt{\frac{\tilde{G}_T}{\tilde{C}_T}}} = \frac{16}{\tilde{C}_T\pi^3}\sum_{m=1}^{\infty}\frac{\tanh a\tilde{\alpha}_m}{(2m-1)^3\frac{a}{h}\sqrt{\frac{\tilde{G}_T}{\tilde{C}_T}}} \tag{40}$$

$$\tilde{\zeta} = \tilde{\lambda}\frac{\tilde{C}_T^2-\tilde{C}_{23}^2}{\tilde{C}_T-\tilde{\lambda}\tilde{C}_{23}^2} = \frac{\tilde{\lambda}\tilde{C}_T(1+\tilde{\nu}_T)(1-\tilde{\nu}_T-2\tilde{\nu}_{LT}\tilde{\nu}_{TL})}{(1-\tilde{\nu}_{LT}\tilde{\nu}_{TL})^2-\tilde{\lambda}\tilde{C}_T(\tilde{\nu}_T+\tilde{\nu}_{LT}\tilde{\nu}_{TL})^2}. \tag{41}$$

In addition, the CP gives the Laplace transform of the averaged laminate axial modulus as:

$$\tilde{E}_x^c = \frac{\tilde{H}_{11}\tilde{H}_{22}-\tilde{H}_{12}^2}{\tilde{H}_{22}(q+r)}. \tag{42}$$

Recall that one of the steps involved in implementing the CP and arriving at the relationships given above was the substitution of the Carson transform of certain viscoelastic material properties for their elastic counterparts. It is noted that there are only four fundamental material properties involved in eqns (34)–(42): E_L, E_T, ν_T, and ν_{LT}. Given the Carson transform of these four, the Carson transforms of C_L, C_T, etc., and α_n are given by eqns (36) and (37). These in turn are used in determining the Carson transforms of the Q's, H's, λ, and ζ; and ultimately of Υ, Ψ, and Ω. This is accomplished according to eqns (38)–(41) and (35).

The CP, as outlined above, is conceptually straightforward. If the viscoelastic lamina properties $E_L(t)$, $E_T(t)$, $\nu_{LT}(t)$, and $\nu_T(t)$ are known and can be Carson transformed, then the Laplace transform of the viscoelastic solution to the problem of uniaxial enforced displacement is given by eqns (34)–(42). We reiterate that this method is restricted to the case of fixed cracks. The finite element method, discussed later, is burdoned by no such restriction. Since the viscoelastic material properties $E_L(t)$, $E_T(t)$, $\nu_{LT}(t)$, and $\nu_T(t)$ are the building blocks for the analysis, we next turn our attention to their determination.

Viscoelastic material properties. There are at least two methods available for determining the viscoelastic lamina properties needed in the analysis. One is to extract them 'directly' from experimental creep or relaxation tests of laminated test specimens. A second method is to combine the properties of the fiber and the matrix in a micromechanical model to extract estimates of the lamina properties. Since an insufficient amount of experimental data (for candidate HSCT material systems) has been presented in the literature for the authors to apply the first approach, it is the second method that is used here.

The micromechanical model chosen for this study is that of Hashin and Rosen (1964) and is commonly referred to as the Composite Cylinders Assemblage (CCA) model, or simply the composite cylinders model. This model, originally written for elastic composites, was later expressed in the frequency (Fourier) domain by Hashin (1970) in order to produce complex moduli of viscoelastic composites. The expressions given by Hashin (1970) are reformulated here in an s (Laplace) domain. It should be noted that the form of the equations given below represents a special case of the model for which v_M (the subscript M is used to denote matrix and the subscript f to denote fiber) along with all fiber properties are assumed to be constant in time. Both fiber and matrix properties are assumed to be isotropic. With these assumptions made, the viscoelastic micromechanics model results in the following Carson transformed ply properties:

$$\begin{aligned}\tilde{K}_M &= \left[\frac{1}{3(1-2v_M)}\right]\tilde{E}_M \\ \tilde{\mu}_M &= \left[\frac{1}{2(1+v_M)}\right]\tilde{E}_M \\ \tilde{k}_M &= \left[\frac{1}{2(1+v_M)(1-2v_M)}\right]\tilde{E}_M \end{aligned} \tag{43}$$

$$K_f = \frac{E_f}{3(1-2v_f)} \quad \mu_f = \frac{3K_f E_f}{9K_f - E_f} \quad k_f = \frac{9K_f^2}{9K_f - E_f} \tag{44}$$

$$\tilde{k} = \frac{\tilde{k}_M(k_f + \tilde{\mu}_M)V_M + k_f(\tilde{k}_M + \tilde{\mu}_M)V_f}{(k_f + \tilde{\mu}_M)V_M + (\tilde{k}_M + \tilde{\mu}_M)V_f} \tag{45}$$

$$\tilde{G}_{LT} = \tilde{\mu}_M\left[\frac{\tilde{\mu}_M V_M + \mu_f(1+V_f)}{\tilde{\mu}_M(1+V_f) + \mu_f V_M}\right] \tag{46}$$

$$\tilde{E}_L = \tilde{E}_M V_M + E_f V_f + \frac{4V_M V_f(v_f - v_M)^2}{\dfrac{V_M}{k_f} + \dfrac{V_f}{\tilde{k}_M} + \dfrac{1}{\tilde{\mu}_M}} \tag{47}$$

$$\tilde{v}_{LT} = v_M V_M + v_f V_f + \frac{V_M V_f(v_f - v_M)\left(\dfrac{1}{\tilde{k}_M} - \dfrac{1}{k_f}\right)}{\dfrac{V_M}{k_f} + \dfrac{V_f}{\tilde{k}_M} + \dfrac{1}{\tilde{\mu}_M}} \tag{48}$$

$$\tilde{G}_T = \frac{\tilde{\mu}_M(1+\alpha V_f^3)(\rho + \beta_M V_f) - 3V_f V_M^2 \beta_M^2}{(1+\alpha V_f^3)(\rho - V_f) - 3V_f V_M^2 \beta_M^2} \tag{49}$$

$$\tilde{E}_T = \frac{4\tilde{k}\tilde{G}_T}{\tilde{k} + m\tilde{G}_T} \tag{50}$$

$$\tilde{v}_T = \frac{\tilde{E}_T - 2\tilde{G}_T}{2\tilde{G}_T} \tag{51}$$

Table 1. Fiber and matrix properties of IM7/8320

$E_f = 275{,}788\ \text{MPa}$
$\nu_f = 0.2$
$V_f = 0.6$
$\tilde{E}_M = \dfrac{6.8947 \times 10^6}{880 + 19.7805\Gamma(1.33)s^{-0.33}}\ \text{MPa}$
$\nu_M = 0.3$
$V_M = 0.4$

where

$$\beta_M = \frac{1}{3-4\nu_M} \quad \beta_f = \frac{1}{3-4\nu_f} \quad \gamma = \frac{\mu_f}{\mu_M} = \frac{2(1+\nu_M)\mu_f}{\tilde{E}_M} \tag{52}$$

$$\alpha = \frac{\beta_M - \gamma\beta_f}{1+\beta_f} \quad \rho = \frac{\gamma+\beta_M}{\gamma-1} \quad m = 1 + \frac{4\tilde{k}\tilde{\nu}_{LT}^2}{\tilde{E}_L}. \tag{53}$$

In the above, K represents the bulk modulus, k the plane-strain bulk modulus, and μ the shear modulus. The symbols V_f and V_M are used to denote volume fractions. Equations (47), (50), (48), and (51) provide the Carson transform of all ply properties needed. These may be substituted directly into the CP as given in eqns (34)–(42).

The material properties used herein for the fiber and matrix are given in Table 1. These properties are representative of the thermoplastic IM7/8320 which is one of the material systems being considered for HSCT structure. The fiber (IM7) is an intermediate modulus polyacrylonitrile (PAN) based fiber manufactured by Hercules. The matrix (8320) is a polysulfone manufactured by AMOCO and is sometimes referred to as RadelX. The glass transition temperature, T_g, for the material system was found by Gates and Feldman (1993) to be 221.3°C.

The values of ν_f and ν_M given in Table 1 are estimates. The expression for $\tilde{E}_M$ given in Table 1 is derived from the creep compliance master curve of S_{22} given by Gates and Feldman (1993) for IM7/8320 at 195°C and loaded to 3.04 MPa. Recall that 195°C is representative of expected skin temperatures in an HSCT operating at Mach 2.4.

Illustrative example problem. Consider a $[0/90_2]_s$ laminate with crack spacing $a/h = 5$, loaded by an enforced displacement of $u_0/a = 0.001H(t)$ while Θ is zero. Here $H(t)$ denotes the Heaviside step function. Suppose that we wish to determine the $\sigma_{xx}(\chi_k, t)$ component of stress at the origin of the coordinate system shown in Fig. 2 along with the averaged laminate axial modulus, $E_x^c(t)$. Note that the state of stress at this point is of particular interest because it is the location at which the next matrix crack is most likely to form.

To do this, the fiber and matrix properties given in Table 1 are used in Hashin's micromechanics model (eqns (43)–(53)) to calculate the Carson transform of the lamina properties $E_L(t)$, $E_T(t)$, $\nu_{LT}(t)$, and $\nu_T(t)$. These are then used in the viscoelastic correspondence principle (eqns (34)–(41)) to determine the Laplace transform of the $\sigma_{xx}(\chi_k, t)$ component of stress. They are similarly used in eqn (42) to determine the Laplace transform of $E_x^c(t)$. Finally, Schapery's (1962) method of approximate Laplace transform inversion is used to calculate $\sigma_{xx}(\chi_k, t)$ and $E_x^c(t)$. Schapery's method of approximate Laplace transform inversion is repeated here for completeness. It is given by:

$$\psi(t) \approx \psi(s)|_{s=0.56/t}. \tag{54}$$

The accuracy of this approximation will be addressed later in this paper. Results generated from the calculations just described will also be given later where they will be compared to numerical prediction.

NUMERICAL ANALYSIS

The illustrative example of the previous section will now be solved numerically. In order to conduct the numerical analysis, the authors employ the finite element code ORTHO3D. This program, developed by the authors, is a general purpose three dimensional code suitable for the solution of uncoupled thermoviscoelastic IBVP's involving nonaging, orthotropic, thermorheologically simple materials. Although a detailed description of the finite element formulation and of ORTHO3D will not be included here (the interested reader is referred to Zocher (1995), Zocher *et al.* (1997) for these), we shall provide a brief outline of the numerical method.

Numerical method

The constitutive equations for a possibly nonhomogeneous, nonaging orthotropic thermorheologically simple material can be expressed in the form given by Schapery (1984) as follows:

$$\sigma_{ij}(\chi,\xi) = \int_0^{\xi} C_{ijkl}(\xi-\xi')\frac{\partial\varepsilon_{kl}(\chi,\xi')}{\partial\xi'}\,d\xi' - \int_0^{\xi}\beta_{ij}(\xi-\xi')\frac{\partial\Theta(\chi,\xi')}{\partial\xi'}\,d\xi'. \tag{55}$$

In the above, C_{ijkl} is the fourth order tensor of relaxation moduli relating stress to mechanical strain, β_{ij} is the second order tensor of relaxation moduli relating stress to thermal strain, and ξ is the reduced time which is defined as:

$$\xi = \xi(t) \equiv \int_0^t \frac{1}{a_T}\,d\tau \quad \xi' = \xi(t') \equiv \int_0^{t'}\frac{1}{a_T}\,d\tau \tag{56}$$

with

$$a_T = a_T(T(\tau)) \quad \text{or equivalently,} \quad a_T = a_T(\Theta(\tau)). \tag{57}$$

The term a_T is the shift factor of the time-temperature superposition principle. The shift factor is essentially a material property; it will often be expressed in terms of an Arrhenius relation or the familiar WLF formula. The symbol $\equiv$ is used herein to mean "is defined as".

If we: (1) subdivide the time line (reduced time) into discrete intervals such that $\xi_{n+1} = \xi_n + \Delta\xi$, (2) assume that the state of stress is known at time ξ_n, (3) express the relaxation moduli C_{ijkl} and β_{ij} in terms of Dirichlet–Prony series, and (4) assume that the variation in ε_{kl} and Θ is linear over each $\Delta\xi$, then we can convert the constitutive equations given in eqn (55) into an algebraic incremental form which is given by:

$$\Delta\sigma_{ij} = C'_{ijkl}\,\Delta\varepsilon_{kl} - \beta'_{ij}\,\Delta\Theta + \Delta\sigma^R_{ij} \tag{58}$$

where C'_{ijkl}, β'_{ij}, $\Delta\varepsilon_{kl}$, and $\Delta\Theta$ are given by:

$$C'_{ijkl} \equiv C_{ijkl_\infty} + \frac{1}{\Delta\xi}\sum_{m=1}^{M_{ijkl}}\eta_{ijkl_m}(1-e^{-\Delta\xi/\rho_{ijkl_m}}) \quad (\text{no sum on} \quad i,j,k,l) \tag{59}$$

$$\beta'_{ij} \equiv \beta_{ij_\infty} + \frac{1}{\Delta\xi}\sum_{p=1}^{P_{ij}}\eta_{ij_p}(1-e^{-\Delta\xi/\rho_{ij_p}}) \quad (\text{no sum on} \quad i,j) \tag{60}$$

$$\Delta\varepsilon_{kl} \equiv \dot{\varepsilon}_{kl}\,\Delta\xi \quad \Delta\Theta \equiv \dot{\Theta}\,\Delta\xi \tag{61}$$

and $\Delta\sigma_{ij}^R$ is given by:

$$\Delta\sigma_{ij}^R = -\sum_{k=1}^{3}\sum_{l=1}^{3} A_{ijkl} + \sum_{p=1}^{P_{ij}} (1-\mathrm{e}^{-\Delta\xi/\rho_{ij_p}})\beta_{ij_p}(\xi_n) \quad \text{(no sum on} \quad i,j) \tag{62}$$

where

$$A_{ijkl} = \sum_{m=1}^{M_{ijkl}} (1-\mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}})S_{ijkl_m}(\xi_n) \quad \text{(no sum on} \quad i,j,k,l) \tag{63}$$

$$S_{ijkl_m}(\xi_n) = \mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}}S_{ijkl_m}(\xi_n-\Delta\xi)+\eta_{ijkl_m}\dot{\varepsilon}_{kl}(1-\mathrm{e}^{-\Delta\xi/\rho_{ijkl_m}}) \quad \text{(no sum on} \quad i,j,k,l) \tag{64}$$

$$B_{ij_p}(\xi_n) = \mathrm{e}^{-\Delta\xi/\rho_{ij_p}}B_{ij_p}(\xi_n-\Delta\xi)+\eta_{ij_p}\dot{\Theta}(1-\mathrm{e}^{-\Delta\xi/\rho_{ij_p}}) \quad \text{(no sum on} \quad i,j). \tag{65}$$

In the above, the η's and the ρ's are dashpot coefficients and relaxation times, respectively, for the Dirichlet–Prony series (in this case Wiechert models) for the relaxation moduli, and the M's and P's are the number of spring/dashpot pairs in the Wiechert model for a given relaxation modulus. The reader is urged to read eqns (59)–(65) carefully so as to avoid confusion. Note that some ρ's and η's possess two subscripts while others possess four. Those with four subscripts are associated with the Wiechert model of a particular member of C_{ijkl} whereas those with two subscripts are associated with the Wiechert model of particular member of β_{ij}. Hence the four-subscripted ρ's and η's are distinct from the two-subscripted variety. The use of ρ and η to represent relaxation times and dashpot coefficients for the Wiechert models of both C_{ijkl} and β_{ij} is admittedly potentially confusing, but should cause the reader no undue burden with the foregoing note of caution. Note that the S's and B's are determined in a recursive fashion.

The incrementalized constitutive formula given in eqn (58) is central to the development of the finite element formulation. Applying the method of weighted residuals, the governing partial differential equation which is given in eqn (29) can be converted to the symmetric variational form given by:

$$\int_\Omega \sigma_{ji}\varepsilon_{ij}\,\mathrm{d}V = \int_\Omega \rho f_i v_i\,\mathrm{d}V + \int_{\partial\Omega_2} T_i v_i\,\mathrm{d}S \tag{66}$$

where v_i is an arbitrary admissible test function (in this case test displacement) and $\varepsilon_{ij} = 1/2(v_{i,j}+v_{j,i})$. Equation (66), evaluated at time ξ_{n+1} (remember that we assume that the solution is known at time ξ_n) is given by:

$$\int_\Omega \sigma_{ji}^{n+1}\varepsilon_{ij}^{n+1}\,\mathrm{d}V = \int_\Omega \rho f_i^{n+1}v_i^{n+1}\,\mathrm{d}V + \int_{\partial\Omega_2} T_i^{n+1}v_i^{n+1}\,\mathrm{d}S \tag{67}$$

where the superscript "$n+1$" denotes "at reduced time ξ_{n+1}". Since the stress–strain relations (eqn (58)) are incrementalized, it is necessary to incrementalize eqn (67). Let us define the following:

$$\begin{aligned}
\Delta\sigma_{ji} &\equiv \sigma_{ji}^{n+1}-\sigma_{ji}^{n} \Rightarrow \sigma_{ji}^{n+1} = \sigma_{ji}^{n}+\Delta\sigma_{ji} \\
\Delta\varepsilon_{ij} &\equiv \varepsilon_{ij}^{n+1}-\varepsilon_{ij}^{n} \Rightarrow \varepsilon_{ij}^{n+1} = \varepsilon_{ij}^{n}+\Delta\varepsilon_{ij} \\
\Delta u_i &\equiv u_i^{n+1}-u_i^{n} \Rightarrow u_i^{n+1} = u_i^{n}+\Delta u_i \\
\Delta v_i &\equiv v_i^{n+1}-v_i^{n} \Rightarrow v_i^{n+1} = v_i^{n}+\Delta v_i.
\end{aligned} \tag{68}$$

Now recognizing that v_i^n and ε_{ij}^n are zero (a consequence of u_i^n being known), substitution

Table 2. Wiechert models for IM7/8320 at 195°C

m	C_{1111}: C_{ijkl_m}	C_{1111}: η_{ijkl_m}	C_{2222}, C_{3333}: C_{ijkl_m}	C_{2222}, C_{3333}: η_{ijkl_m}
∞	0.16582E03	—	0.14030E01	—
1	0.28538E−01	0.57074E−03	0.97437E−01	0.19487E−02
2	0.58277E−01	0.11656E−01	0.20560E00	0.41120E−01
3	0.12555E00	0.25109E00	0.43423E00	0.86847E00
4	0.25571E00	0.51141E01	0.89846E00	0.17969E02
5	0.51050E00	0.10210E03	0.18009E01	0.36018E03
6	0.91425E00	0.18286E04	0.32963E01	0.65927E04
7	0.13780E01	0.27561E05	0.51295E01	0.10259E06
8	0.15251E01	0.30501E06	0.58575E01	0.11715E07
9	0.12850E01	0.25700E07	0.50730E01	0.10146E08
10	0.75091E00	0.15018E08	0.30053E01	0.60106E08
11	0.58050E00	0.11610E09	0.23489E01	0.46978E09

m	C_{2323}: C_{ijkl_m}	C_{2323}: η_{ijkl_m}	C_{1313}, C_{1212}: C_{ijkl_m}	C_{1313}, C_{1212}: η_{ijkl_m}
∞	0.42113E00	—	0.50665E00	—
1	0.23349E−01	0.46697E−03	0.38076E−01	0.76153E−03
2	0.49173E−01	0.98347E−02	0.79876E−01	0.15976E−01
3	0.10439E00	0.20877E00	0.16897E00	0.33794E00
4	0.21797E00	0.43592E01	0.34793E00	0.69589E01
5	0.44702E00	0.89405E02	0.69320E00	0.13864E03
6	0.84916E00	0.16984E04	0.12537E01	0.25074E04
7	0.13936E01	0.27873E05	0.19163E01	0.38325E05
8	0.16628E01	0.33256E06	0.21556E01	0.43112E06
9	0.14825E01	0.29650E07	0.18484E01	0.36968E07
10	0.89019E00	0.17804E08	0.10900E01	0.21800E08
11	0.70224E00	0.14045E09	0.84930E00	0.16987E09

m	C_{1122}, C_{1133}: C_{ijkl_m}	C_{1122}, C_{1133}: η_{ijkl_m}	C_{2233}: C_{ijkl_m}	C_{2233}: η_{ijkl_m}
∞	0.45611E00	—	0.56071E00	—
1	0.34537E−01	0.69072E−03	0.49751E−01	0.99499E−03
2	0.72871E−01	0.14574E−01	0.11001E00	0.22003E−01
3	0.15378E00	0.30756E00	0.21930E00	0.43859E00
4	0.31676E00	0.63352E01	0.47598E00	0.95197E01
5	0.63079E00	0.12616E03	0.87578E00	0.17515E03
6	0.11397E01	0.22794E04	0.16682E01	0.33363E04
7	0.17388E01	0.34777E05	0.22099E01	0.44198E05
8	0.19512E01	0.39024E06	0.26167E01	0.52335E06
9	0.16692E01	0.33383E07	0.21071E01	0.42142E07
10	0.98292E00	0.19659E08	0.12249E01	0.24498E08
11	0.76505E00	0.15300E09	0.94438E00	0.18887E09

of eqn (68) into eqn (67) and following the usual finite element development produces a set of algebraic equations of the form shown below which can be solved to produce an incremental solution to the thermoviscoelastic IBVP.

$$[K]\{\Delta u\} = \{F\}. \tag{69}$$

The finite element program (ORTHO3D) based on the foregoing outline has been verified through the solution of a large number of test problems for which accepted analytical solutions are available (Zocher (1995)). Unfortunately, space does not permit the inclusion of those results here.

Illustrative example

In order to conduct the numerical analysis, we must first determine the Wiechert models (i.e., C_{ijkl}) for the material system IM7/8320 at 195°C. This is accomplished using the micromechanical model of Hashin discussed in the previous section. The resultant Wiechert models for the orthotropic material system are given in Table 2. The units on the C_{ijkl_m} and η_{ijkl_m} of Table 2 are GPa and GPa-s, respectively.

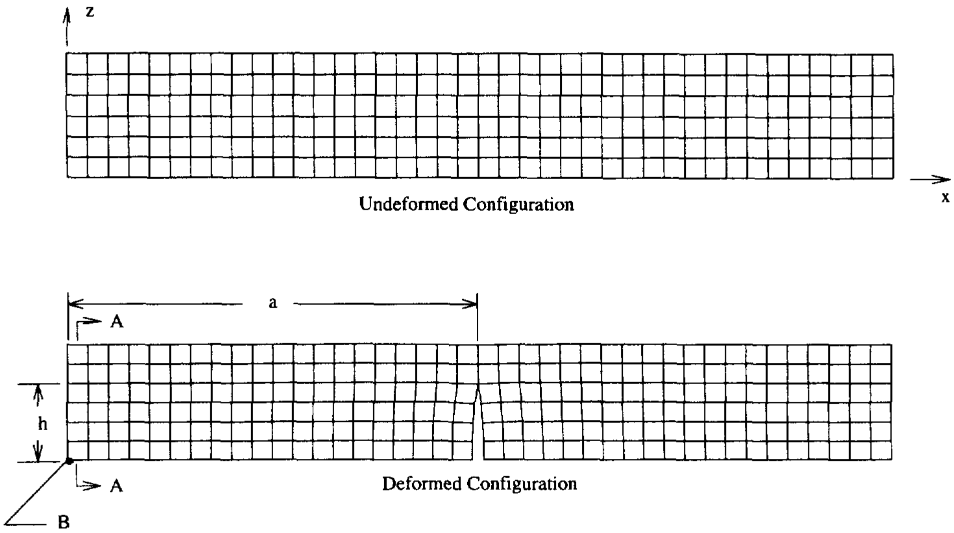

Fig. 4. Reference and deformed finite element mesh of one-quarter laminate.

The finite element mesh used in the analysis is shown in Fig. 4 in both the reference and deformed configurations. Note that symmetry conditions were exploited so that the entire laminate does not have to be modelled. This mesh is clearly not fine enough to provide an accurate assessment of the crack-tip stress field. It is, however, sufficiently fine for the purpose at hand: the determination of the state of stress at point B (Fig. 4) and the averaged laminate axial modulus.

The averaged laminate axial modulus was extracted from the finite element results as follows. The stresses on a cutting plane (such as plane A–A shown in Fig. 4) were used to determine a resultant force, P, that would have to be applied over the plane A–A cross-section of the body in order to maintain equilibrium of the remaining portion of the body, if the portion either to the right or to the left of plane A–A were removed. An average stress was then calculated as P divided by the area of the surface at plane A–A. This average stress was then divided by the average strain (0.001) in the body to yield the averaged laminate axial modulus. The stress at point B was determined through interpolation of the value of the stress at neighboring integration points.

Finite element predictions are given in Fig. 5 where they are compared to the analytical results of the previous section. Recall that the analytical solution of Lee *et al.* produces an upper bound prediction on stress and modulus (lower bound on displacement). Consequently, we should expect the viscoelastic extension of the solution of Lee *et al.* to exhibit upper bound behavior as well. This is, for the most part, what we see in Fig. 5, where the analytical solution lies above the numerical solution for all but long time.

The deviation in upper bound behavior of the analytical solution at long time is not of particular concern since it can be attributed to several approximations. The most important of these errors arises from our use of Schapery's approximate Laplace transform inversion. It can be shown that this method of Laplace transform inversion is highly accurate when the relationship between the quantity being transformed and time is approximately linear in log–log space. An investigation of the lamina properties shows an approximate linear relationship between them and time in log–log space for $\mathrm{LOG}_{10}\, t \in [3, 6]$ and a markedly curvilinear relationship outside this region. This being the case, we can interpret the relationship between the two predictions in Fig. 5 as follows. The difference between the two predictions is initially relatively large due to error in the method of approximate Laplace transform inversion caused by nonlinearity in log–log space of the material properties; this decreases throughout the linear region ($\mathrm{LOG}_{10}\, t \in [3, 6]$), and then grows again once this region is passed. A second source of error resides in the fact that the time step was increased as time progressed in the numerical calculation, becoming relatively large for $\mathrm{LOG}_{10}\, t > 7$. A Δt of 0.01 seconds was used from the start of program execution out to $t = 10$ seconds, for $10 \leqslant t \leqslant 50$ a Δt of 0.1 seconds was used, for $50 \leqslant t \leqslant 100$ a Δt of 1

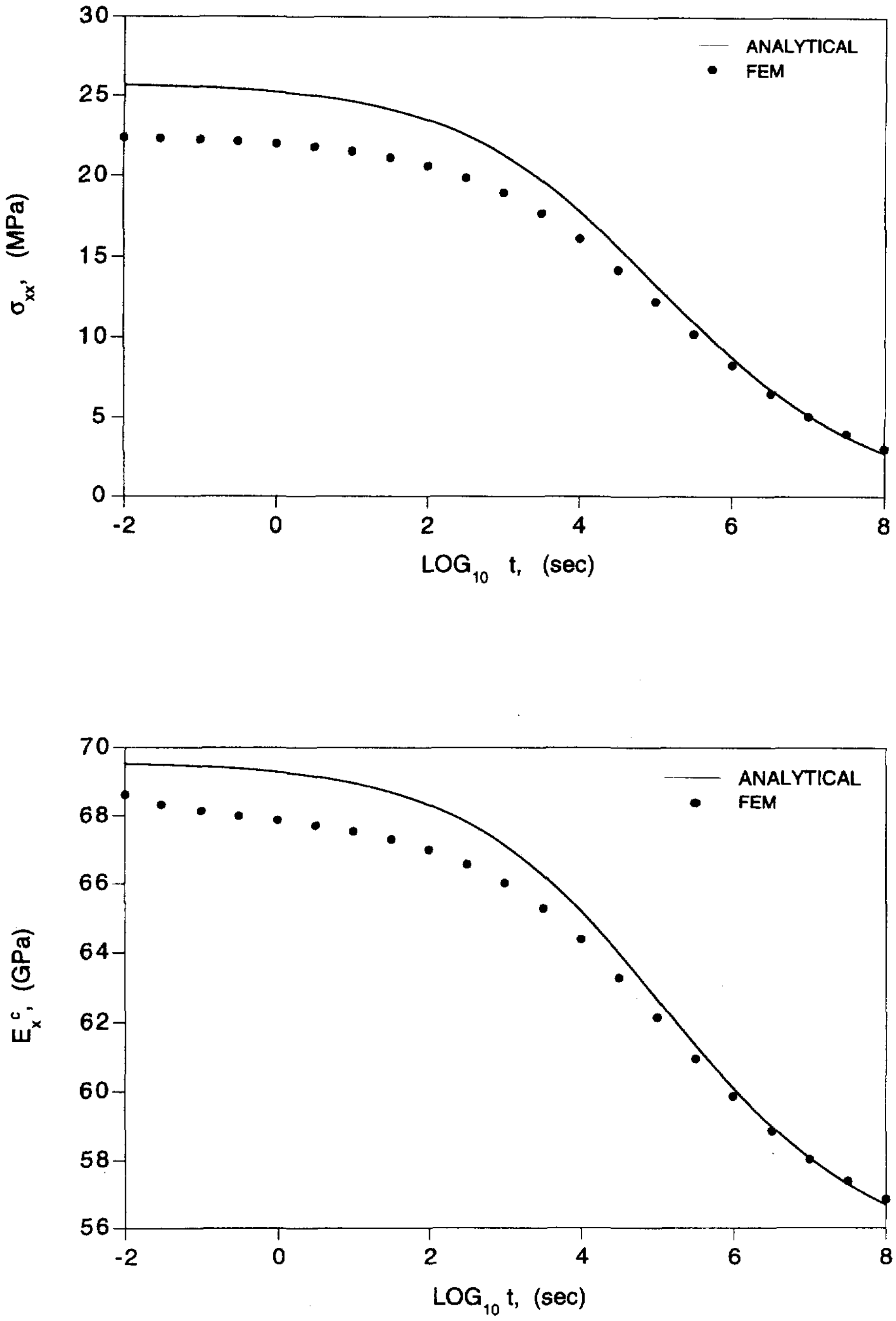

Fig. 5. Analytical and numerical results; matrix-cracked laminate.

second was used, and for each succeeding decade, Δt was increased an order of magnitude becoming equal to 1.E06 for $10^7 \leqslant t \leqslant 10^8$. A third reason for the observed difference between the analytical and numerical predictions derives from differences in the assumed displacement fields. In the finite element method, both kinematics and kinetics must be satisfied, not so in the Ritz method where only the kinematic constraints must be satisfied. The displacement field of Lee *et al.* in fact satisfies equilibrium in the x- and y-directions only. Considering this, along with the error associated with the method of approximate Laplace transform inversion, it is likely (although not proven here) that the finite element result is the more accurate of the two.

Although the term "large" was used at one point in the previous paragraph to describe differences between the two predictions, it was used in a relative sense. Actually the analytical prediction, considering all of the approximations involved, turned out to be quite accurate vis-à-vis the numerical prediction.

CONCLUSIONS

Analytical (CP) and numerical (finite element) methods have been presented for the analysis of matrix-cracked laminates which exhibit time-dependent and damage-dependent behavior. These methods have been demonstrated through the solution of a simple illustrative example problem. To the knowledge of the authors, this work represents some of the first research to be presented on the topic of matrix-cracking in viscoelastic laminates. That being the case, it was important that two distinctly different methods of analysis be used in order to instill confidence in either approach. The fact that the results from the two methods of analysis were in close agreement implies that either approach may prove to be valuable in parametric studies aimed at gaining a better understanding of the effects of matrix cracking in viscoelastic composites.

While the present work involved a single time-independent damage state only, it must be emphasized that the results produced in an analysis of this type represent the essential first step in a prediction of damage evolution. The authors view damage evolution as the next logical step for this research. The ultimate aim of our effort is to develop the capability of predicting component life. We emphasize that only the finite element method can be used to predict damage evolution; the correspondence principle is restricted to fixed crack geometry.

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